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# Generalizations of the Virasoro algebra and matrix Sturm–Liouville operators

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**Abstract**

We study the series of Lie algebras generalizing the Virasoro algebra introduced in [V. Yu, Ovsienko, C. Roger, Functional Anal. Appl. 30 (4) (1996)]. We show that the coadjoint representation of each of these Lie algebras has a natural geometrical interpretation by matrix differential operators generalizing the Sturm–Liouville operators. 

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## 1. Introduction

The starting point of our work is a deep relation between the Virasoro algebra and the space of Strum–Liouville operators remarked by Kirillov [5,6] and Segal [14]. Space of Sturm–Liouville operators gives a natural realization for the coadjoint representation of the Virasoro algebra. This important relation leads to the classification of the coadjoint orbits of the Virasoro algebra [5]. Moreover, it is well known that the Korteweg–de Vries equation appeared as the Euler equation of the Virasoro–Bott group [10,15], which increased the interest of the coadjoint action of the Virasoro algebra.

In this paper, we extend the Kirillov–Segal result to Lie algebras generalizing the Virasoro algebra: we present relations between these algebras and Sturm–Liouville type matrix operators and we give a realization for the coadjoint representation of these Lie algebras. The corresponding Korteweg–de Vries-type equations will be studied in a subsequent paper.

These results are obtained by a method due to Kirillov [5], using the Neveu–Schwarz and Ramond superalgebras generalizing the Virasoro algebra. The Lie superalgebras generaliz-


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ing the extensions of the Virasoro algebra studied in the present paper have been classified in [9]; this paper is a natural continuation of [9].

It is worth noticing that an analogous approach has been applied in [7] to another class of Lie superalgebras generalizing the Virasoro algebra, so-called stringy superalgebras.

### 1.1. *Virasoro algebra*

Let *Vect*$(S^1)$ be the Lie algebra of smooth vector field on $S^1 : f = f(x)(\mathrm{d}/\mathrm{d}x)$, where $f(x+2\pi) = f(x)$, with the commutator

$$\left[f(x)\frac{\mathrm{d}}{\mathrm{d}x}, g(x)\frac{\mathrm{d}}{\mathrm{d}x}\right] = (f(x)g'(x) - f'(x)g(x))\frac{\mathrm{d}}{\mathrm{d}x}.$$

The *Virasoro algebra* is the unique (up to isomorphism) non-trivial central extension of *Vect*$(S^1)$. It is given by the Gelfand–Fuchs cocycle:

$$c\left(f(x)\frac{\mathrm{d}}{\mathrm{d}x}, g(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) = \int_{S^1} f'(x)g''(x)\,\mathrm{d}x. \tag{1}$$

Denote by *Vir* the Virasoro algebra.

### 1.2. *Space of Sturm–Liouville operators as a Vect*$(S^1)$*-module*

Consider the space of *Sturm–Liouville operators*:

$$L = -2c\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u(x), \tag{2}$$

where $c \in \mathbf{R}$ and $u$ is a periodic potential: $u(x+2\pi) = u(x) \in C^\infty(\mathbf{R})$.

Following classicial works we define a natural *Vect*$(S^1)$-module structure on the space of Sturm–Liouville operators.

Let $\mathcal{F}_\lambda$ be the space of all tensor densities on $S^1$ of degree $\lambda : a = a(x)(\mathrm{d}x)^\lambda$. The Lie algebra *Vect*$(S^1)$ *acts* on $\mathcal{F}_\lambda$ by the Lie derivative

$$L^{(\lambda)}_{f(x)(\mathrm{d}/\mathrm{d}x)}a = (f(x)a'(x) + \lambda f'(x)a(x))(\mathrm{d}x)^\lambda. \tag{3}$$

Let us consider the Sturm–Liouville operators as operators from $\mathcal{F}_{-1/2}$ to $\mathcal{F}_{3/2}$:

$$L : \mathcal{F}_{-1/2} \to \mathcal{F}_{3/2}.$$

**Definition 1.1** (cf. [6,16]). The action of a vector field $f(\mathrm{d}/\mathrm{d}x)$ on a Sturm–Liouville operator $L$ is given by the commutator

$$T_{(f\,\mathrm{d}/\mathrm{d}x)}(L) = L^{(3/2)}_{f\,\mathrm{d}/\mathrm{d}x} \circ L - L \circ L^{(-1/2)}_{f\,\mathrm{d}/\mathrm{d}x}. \tag{4}$$

The result of the action (4) is a scalar operator of multiplication by a function

$$T_{(f\,\mathrm{d}/\mathrm{d}x)}(L) = fu' + 2f'u - cf'''.$$

Note that a scalar operator is a Sturm–Liouville operator with $c = 0$, and therefore, the space of Sturm–Liouville operators is indeed a $Vect(S^1)$-module.

### 1.3. Coadjoint representation of the Virasoro algebra

Following Kirillov [6], we consider the *regular part*, $Vir^*_{reg}$, of the dual space to the Virasoro algebra. $Vir^*_{reg}$ can be identified to the space $\mathcal{F}_2 \oplus \mathbf{R}$. The pairing $\langle\, ,\rangle : Vir^*_{reg} \otimes Vir \to \mathbf{R}$ is

$$\left\langle \begin{pmatrix} u(x)\,\mathrm{d}x^2 \\ c_1 \end{pmatrix}, \begin{pmatrix} f\,\frac{\mathrm{d}}{\mathrm{d}x} \\ r \end{pmatrix} \right\rangle = \int_0^{2\pi} u(x) f(x)\,\mathrm{d}x + c_1 r.$$

The following fact shows that the $Vect(S^1)$-module of Sturm–Liouville operators is a natural realization of the coadjoint representation of the Virasoro algebra [5,6,14].

**Theorem 1.2** (Kirillov [6], Segal [14]). *The coadjoint action of the Virasoro algebra on* $Vir^*_{reg}$ *coincides with the* $Vect(S^1)$*-action* (4).

**Proof.** From the definition of the coadjoint action:

$$\left\langle ad^*_{\begin{pmatrix} f\,\mathrm{d}/\mathrm{d}x \\ r \end{pmatrix}} \begin{pmatrix} u(\mathrm{d}x)^2 \\ c_1 \end{pmatrix}, \begin{pmatrix} g\,\frac{\mathrm{d}}{\mathrm{d}x} \\ s \end{pmatrix} \right\rangle$$
$$= -\left\langle \begin{pmatrix} u(\mathrm{d}x)^2 \\ c_1 \end{pmatrix}, \left[ \begin{pmatrix} f\,\frac{\mathrm{d}}{\mathrm{d}x} \\ r \end{pmatrix}, \begin{pmatrix} g\,\frac{\mathrm{d}}{\mathrm{d}x} \\ s \end{pmatrix} \right] \right\rangle,$$

one obtains by direct calculation

$$ad^*_{\begin{pmatrix} f\,\mathrm{d}/\mathrm{d}x \\ r \end{pmatrix}} \begin{pmatrix} u(\mathrm{d}x)^2 \\ c_1 \end{pmatrix} = \begin{pmatrix} (fu' + 2f'u - c_1 f''')(\mathrm{d}x)^2 \\ 0 \end{pmatrix}.$$

□

**Remark 1.3.** *Note that the coadjoint action of the Virasoro algebra is in fact a* $Vect(S^1)$*-action* (*the center acts trivially*).

## 2. The Ovsienko–Roger algebras

Let us consider a series of nine Lie algebras generalizing the Virasoro algebra. These Lie algebras are defined as *extensions* of the Lie algebra $Vect(S^1)$ by the modules of tensor densities on $S^1$. These extensions have been introduced by Ovsienko and Roger in [11] (see also [12]). Let us recall here the main definitions (see [11] for more details).

### 2.1. Central extension of the semi-direct product $Vect(S^1) \ltimes \mathcal{F}_\lambda$

Let us first recall the classification of the central extension of $Vect(S^1) \ltimes \mathcal{F}_\lambda$. These extensions exists if and only if $\lambda = 0$ or 1, cf. [11]. We have

$$H^2(Vect(S^1) \ltimes \mathcal{F}_\lambda; \mathbf{R}) = \begin{cases} \mathbf{R}^3 & \text{for } \lambda = 0, 1, \\ 0 & \text{for } \lambda \neq 0, 1. \end{cases}$$

Each of the algebras $Vect(S^1) \ltimes \mathcal{F}_0$ and $Vect(S^1) \ltimes \mathcal{F}_1$ has a three-dimensional non-trivial central extension given by the following 2-cocycles.

- For both algebras, the continuation of the Gelfand–Fuchs cocycle (1):

$$\tilde{c}((f, a), (g, b)) = \int_{S^1} f'(x) g''(x)\, \mathrm{d}x. \tag{5}$$

Notice that $\tilde{c}$ does not depend on $a$ and $b$.

- Two more non-trivial 2-cocycles in each case:

(a) For $\lambda = 0$,

$$\sigma_1\left(\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a(x)\right), \left(g\frac{\mathrm{d}}{\mathrm{d}x}, b(x)\right)\right) = \int_{S^1} (f''(x)b(x) - g''(x)a(x))\, \mathrm{d}x,$$

$$\sigma_2\left(\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a(x)\right), \left(g\frac{\mathrm{d}}{\mathrm{d}x}, b(x)\right)\right) = \int_{S^1} (a(x)b'(x) - b(x)a'(x))\, \mathrm{d}x.$$

(b) For $\lambda = 1$,

$$\sigma_3\left(\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a\, \mathrm{d}x\right), \left(g\frac{\mathrm{d}}{\mathrm{d}x}, b\, \mathrm{d}x\right)\right) = \int_{S^1} (f'(x)b(x) - g'(x)a(x))\, \mathrm{d}x,$$

$$\sigma_4\left(\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a\, \mathrm{d}x\right), \left(g\frac{\mathrm{d}}{\mathrm{d}x}, b\, \mathrm{d}x\right)\right) = \int_{S^1} (f(x)b(x) - g(x)a(x))\, \mathrm{d}x.$$

**Notation 2.1.** Denote, respectively, by $\mathcal{A}_1$ and $\mathcal{A}_2$ the three-dimensional extensions of $Vect(S^1) \ltimes \mathcal{F}_0$ and $Vect(S^1) \ltimes \mathcal{F}_1$.

### 2.2. Extensions of $Vect(S^1)$ by $\mathcal{F}_\lambda$

Consider the Lie algebras given by extensions of $Vect(S^1)$ with coefficients in $Vect(S^1)$-modules of tensor-densities. The classification of these non-trivial extensions is given by the following result [2]:

$$H^2(Vect(S^1); \mathcal{F}_\lambda) = \begin{cases} \mathbf{R}^2 & \text{for } \lambda = 0, 1, 2, \\ \mathbf{R} & \text{for } \lambda = 5, 7, \\ 0 & \text{for } \lambda \neq 0, 1, 2, 5, 7. \end{cases}$$

The corresponding non-trivial cocycles were given in [11].

1. For $\lambda = 0$, $\lambda = 1$, $\lambda = 2$, there are two non-isomorphic non-trivial extensions.

Let us give the 2-cocycles on $Vect(S^1)$ with value in $\mathcal{F}_\lambda$ representing the non-trivial cohomological classes:

- for $\lambda = 0$,

$$c_0(f, g) = \int_{S^1} f'(x)g''(x)\,\mathrm{d}x \,(\text{case of the Virasoro algebra, here } c_0(f, g) \text{ is a constant function on } S^1),$$

$$\bar{c}_0(f, g) = fg' - f'g;$$

- for $\lambda = 1$,

$$c_1(f, g) = (f'g'' - f''g')\,\mathrm{d}x \quad \text{and} \quad \bar{c}_1(f, g) = (fg'' - f''g)\,\mathrm{d}x;$$

- for $\lambda = 2$,

$$c_2(f, g) = (f'g''' - f'''g')\,\mathrm{d}x^2 \quad \text{and} \quad \bar{c}_2(f, g) = (fg''' - f'''g)\,\mathrm{d}x^2.$$

2. For $\lambda = 5$, $\lambda = 7$, there is a unique non-trivial extension, the 2-cocycles are:
   - for $\lambda = 5$,

$$c_5(f, g) = (f'''g^{(\mathrm{IV})} - f^{(\mathrm{IV})}g''')\,\mathrm{d}x^5;$$

   - for $\lambda = 7$,

$$c_7(f, g) = \left(2\begin{vmatrix} f''' & g''' \\ f^{(\mathrm{VI})} & g^{(\mathrm{VI})} \end{vmatrix} - 9\begin{vmatrix} f^{(\mathrm{IV})} & g^{(\mathrm{IV})} \\ f^{(\mathrm{V})} & g^{(\mathrm{V})} \end{vmatrix}\right)(\mathrm{d}x)^7.$$

**Notation 2.2.** Let us denote $\mathcal{G}_i$ as the Lie algebras given by the non-trivial cocycles $c_i$ and $\bar{\mathcal{G}}_i$ the Lie algebra given by the non-trivial cocycles $\bar{c}_i$.

### *2.3. Central extension of the Lie algebras $\mathcal{G}_i$ and $\bar{\mathcal{G}}_i$*

Let us describe now the central extensions of Lie algebras $\mathcal{G}_i$ and $\bar{\mathcal{G}}_i$ [11,12].

1. Each algebra $\mathcal{G}_i$, $\bar{\mathcal{G}}_i$ has a non-trivial central extension given by the 2-cocycles (5).
2. There exists two more non-trivial central extensions:
   - a central extension of Lie algebra $\mathcal{G}_1$ given by the 2-cocycle

$$\sigma_5\left(\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a\mathrm{d}x\right), \left(g\frac{\mathrm{d}}{\mathrm{d}x}, b\,\mathrm{d}x\right)\right) = \int_{S^1} (f'(x)b(x) - g'(x)a(x))\,\mathrm{d}x;$$

   - a central extension of Lie algebra $\bar{\mathcal{G}}_1$ given by the 2-cocycle

$$\sigma_6\left(\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a\,\mathrm{d}x\right), \left(g\frac{\mathrm{d}}{\mathrm{d}x}, b\,\mathrm{d}x\right)\right) = \int_{S^1} (f(x)b(x) - g(x)a(x))\,\mathrm{d}x.$$

**Notation 2.3.** Let us denote $\mathcal{A}_3$ and $\mathcal{A}_4$ as the two-dimensional central extension of $\mathcal{G}_1$ and $\bar{\mathcal{G}}_1$, respectively. Denote $\mathcal{A}_5, \ldots, \mathcal{A}_9$ as the one-dimensional central extension of $\bar{\mathcal{G}}_0, \mathcal{G}_2, \bar{\mathcal{G}}_2, \mathcal{G}_5, \mathcal{G}_7$, respectively.

## 3. Generalization of the Kirillov–Segal result

In this section, we will extend the Kirillov–Segal result to the case of the Lie algebras $\mathcal{A}_i, i = 1, 2, \dots, 8$. It turns out that these algebras are related to some interesting classes of differential operators that we call matrix Sturm–Liouville operators.

### *3.1. Matrix Sturm–Liouville operators*

We will define a space of matrix operators associated to each of the Lie algebras defined in Section 2. These operators can be considered as generalizations of the Sturm–Liouville operators. The constructed spaces of operators give a geometric realization for the coadjoint representation of each algebra $\mathcal{A}_i,\ i = 1, 2, \dots, 8$, analogous to those in the Virasoro case. We will show that in the case of Lie algebra $\mathcal{A}_9$, such a realization does not exist.

**Definition 3.1.** We will consider matrix differential operators of the following form:

$$\mathcal{L} = \begin{pmatrix} -2c_1\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + u(x) + A & t(x)\dfrac{\mathrm{d}}{\mathrm{d}x} + w(x) \\ -t(x)\dfrac{\mathrm{d}}{\mathrm{d}x} + \tilde{w}(x) & m \end{pmatrix}, \tag{6}$$

where $A$ is some scalar linear differential operator and $t, w, \tilde{w}$ are $2\pi$-periodic functions, $m \in \mathbf{R}$.

Let us associate to each of the Lie algebras $\mathcal{A}_i, i = 1, 2, \dots, 8$, a space of operators $\mathcal{L}$ of the above form (6). The explicit form of $\mathcal{L}$ is given in Table 1 in which $c_2, c_3$ are constants, $v = v(x)$ is a $2\pi$-periodic function and $l_8$ is given by

$$l_8 = 14v\,\frac{\mathrm{d}^6}{\mathrm{d}x^6} + 42v'\,\frac{\mathrm{d}^5}{\mathrm{d}x^5} + 45v''\,\frac{\mathrm{d}^4}{\mathrm{d}x^4} + 20v''\,\frac{\mathrm{d}^3}{\mathrm{d}x^3} + 3v^{(\mathrm{IV})}\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}.$$

Table 1

| Lie algebras | Operators $\mathcal{L}$ | | | | |
|---|---|---|---|---|---|
| | $A$ | $t$ | $w$ | $\tilde{w}$ | $m$ |
| $\mathcal{A}_1$ | $0$ | $c_2$ | $v$ | $v$ | $4c_3$ |
| $\mathcal{A}_2$ | $0$ | $c_2$ | $\frac{1}{2}(v' - c_3)$ | $\frac{1}{2}(v' - c_3)$ | $0$ |
| $\mathcal{A}_3$ | $-4v\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 4v'\frac{\mathrm{d}}{\mathrm{d}x}$ | $c_2$ | $\frac{1}{2}v'$ | $\frac{1}{2}v'$ | $0$ |
| $\mathcal{A}_4$ | $-v'$ | $0$ | $\frac{1}{2}(v' - c_2)$ | $\frac{1}{2}(v' - c_2)$ | $0$ |
| $\mathcal{A}_5$ | $v$ | $0$ | $v$ | $v$ | $0$ |
| $\mathcal{A}_6$ | $-2v'\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2v''\frac{\mathrm{d}}{\mathrm{d}x}$ | $v$ | $\frac{3}{2}v'$ | $\frac{1}{2}v'$ | $0$ |
| $\mathcal{A}_7$ | $2v\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2v'\frac{\mathrm{d}}{\mathrm{d}x} + v''$ | $v$ | $\frac{3}{2}v'$ | $\frac{1}{2}v'$ | $0$ |
| $\mathcal{A}_8$ | $l_8$ | $0$ | $\frac{1}{2}v$ | $\frac{1}{2}v$ | $0$ |

**Remark 3.2.** *The case of algebra $\mathcal{A}_1$ and the corresponding space of matrix operators has been considered in* [8].

We will give a generalization of the action (4) from Section 1.2 for each of the defined spaces of operators (6) and the Lie algebras $\mathcal{A}_i$. To define a $Vect(S^1)$-module structure, and more generally, an $\mathcal{A}_i$-module structure on each space of operators (6), we consider, as in Section 1.2, operators $\mathcal{L}$ as acting on tensor densities:

$$\mathcal{L} : \mathcal{F}_{-1/2} \oplus \mathcal{F}_\theta \to \mathcal{F}_{3/2} \oplus \mathcal{F}_\nu, \tag{7}$$

where the value of $\theta$ and $\nu$ depend on the algebra $\mathcal{A}_i$.

We will define actions of Lie algebras $\mathcal{A}_i$ on $\mathcal{F}_{-1/2} \oplus \mathcal{F}_\theta$ and $\mathcal{F}_{3/2} \oplus \mathcal{F}_\nu$.

### 3.2. *Modules of tensor densities*

#### 3.2.1. *Two families of modules over the semi-direct product*

Let us first consider the action of the semi-direct product $Vect(S^1) \ltimes \mathcal{F}_\lambda$ on the space $\mathcal{F}_\mu \oplus \mathcal{F}_\rho$. It turns out that there exists two natural families of $Vect(S^1) \ltimes \mathcal{F}_\lambda$-module, corresponding to $\rho = \mu + \lambda$ and $\rho = \mu + \lambda + 1$.

**Definition 3.3.** $Vect(S^1) \ltimes \mathcal{F}_\lambda$ acts on the space $\mathcal{F}_\mu \oplus \mathcal{F}_{\mu+\lambda}$ as follows:

$$T_{\begin{pmatrix} f\,\mathrm{d}/\mathrm{d}x \\ a(\mathrm{d}x)^\lambda \end{pmatrix}} \begin{pmatrix} \psi(\mathrm{d}x)^\mu \\ \beta(\mathrm{d}x)^{\mu+\lambda} \end{pmatrix} = \begin{pmatrix} L^{(\mu)}_{f\,\mathrm{d}/\mathrm{d}x}\psi \\ L^{(\mu+\lambda)}_{f\,\mathrm{d}/\mathrm{d}x}\beta + k\psi a \end{pmatrix}, \tag{8}$$

where $k$ is a parameter: $k \in \mathbf{R}$.

**Definition 3.4.** $Vect(S^1) \ltimes \mathcal{F}_\lambda$ acts on space $\mathcal{F}_\mu \oplus \mathcal{F}_{\mu+\lambda+1}$ by

$$\mathcal{T}_{\begin{pmatrix} f\,\mathrm{d}/\mathrm{d}x \\ a(\mathrm{d}x)^\lambda \end{pmatrix}} \begin{pmatrix} \psi(\mathrm{d}x)^\mu \\ \beta(\mathrm{d}x)^{\mu+\lambda+1} \end{pmatrix} = \begin{pmatrix} L^{(\mu)}_{f\,\mathrm{d}/\mathrm{d}x}\psi \\ L^{(\mu+\lambda+1)}_{f\,\mathrm{d}/\mathrm{d}x}\beta + k(\mu\psi a' - \lambda\psi' a) \end{pmatrix}. \tag{9}$$

**Remark 3.5.** *The term* $(\mu\psi a' - \lambda\psi' a)$ *is the Poisson* (*or Shouten*) *bracket which we denote as* $J_1(\psi\, \mathrm{d}x^\mu, a\, \mathrm{d}x^\lambda)$, *see e.g.* [4]. *It is easy to see that formulae* (8) and (9) *indeed define a* $Vect(S^1) \ltimes \mathcal{F}_\lambda$*-action.*

#### 3.2.2. *Deformation of the module structures*

Except for the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ whose action is a semi-direct product action, the actions of Ovsienko–Roger algebras on the spaces $\mathcal{F}_\mu \oplus \mathcal{F}_{\mu+\lambda}$ and $\mathcal{F}_\mu \oplus \mathcal{F}_{\mu+\lambda+1}$ are obtained by deformations of the actions (8) and (9). We are looking for $\mathcal{A}_i$-actions in the following form:

(a) *First type*.

$$\hat{T}_{i\begin{pmatrix} f \\ a \end{pmatrix}} \begin{pmatrix} \psi \\ \beta \end{pmatrix} = T_{\begin{pmatrix} f \\ a \end{pmatrix}} \begin{pmatrix} \psi \\ \beta \end{pmatrix} + \begin{pmatrix} 0 \\ \hat{s}(f, \psi) \end{pmatrix}, \tag{8'}$$

(b) *Second type*.

$$\tilde{\mathcal{T}}_{i\binom{f}{a}}\begin{pmatrix}\psi\\ \beta\end{pmatrix}=\mathcal{T}_{\binom{f}{a}}\begin{pmatrix}\psi\\ \beta\end{pmatrix}+\begin{pmatrix}0\\ \tilde{s}(f,\psi)\end{pmatrix}, \qquad (9')$$

where $\hat{s}$ (resp. $\tilde{s}$) is a bilinear mapping from $Vect(S^1)\oplus\mathcal{F}_\mu$ with value in $\mathcal{F}_{\mu+\lambda}$ (resp. in $\mathcal{F}_{\mu+\lambda+1}$). Note that the center of each algebra $\mathcal{A}_i$ acts trivially.

Let us give some details about the cohomologic nature of the mappings $\hat{s}$, the case of $\tilde{s}$ is analogous. Consider a Lie algebra $\mathcal{A}_i$, $i=3,\dots,8$. Denote by $c_{\mathcal{G}}$ the 2-*cocycle* on $Vect(S^1)$ with values in $\mathcal{F}_\lambda$ defining the extension $\mathcal{G}_i$ (or $\bar{\mathcal{G}}_i$). Let us define the following mappings:

- the *cochain* $s$ on $Vect(S^1)$ with values in $Hom(\mathcal{F}_\mu,\mathcal{F}_{\mu+\lambda})$ defined by

$$s(f)(\psi):=\hat{s}(f,\psi),$$

- the 2-*cocycle* $\bar{c}$ on $Vect(S^1)$ with values in $Hom(\mathcal{F}_\mu,\mathcal{F}_{\mu+\lambda})$ defined by

$$\bar{c}(f,g)(\psi):=\psi c_{\mathcal{G}}(f,g).$$

**Remark 3.6.** *In the case of* $\tilde{s}$ *the* 2-*cocycle* $\bar{c}$ *is defined by*

$$\bar{c}(f,g)(\psi):=J_1(\psi,c_{\mathcal{G}}(f,g)).$$

One obtains immediately the following proposition.

**Proposition 3.7.** *Formula* (8′) *defines an action of the Lie algbera* $\mathcal{A}_i$ *if and only if*

$$\mathrm{d}s=-k\bar{c}. \qquad (10)$$

Proposition 3.7 follows directly from usual definitions.

**Remark 3.8.** *Note that if* $s$ *is a solution of* (10) *and* $s_0$ *a* 1-*cocycle on* $Vect(S^1)$ *with values in* $Hom(\mathcal{F}_\mu,\mathcal{F}_{\mu+\lambda})$, $s+s_0$ *is also a solution of* (10).

*3.2.3. Actions of Lie algebras* $\mathcal{A}_i$

Let us first precise, for each algebra $\mathcal{A}_i$, the spaces of tensor densities considered in (7).

- In the case of algebras $\mathcal{A}_1,\dots,\mathcal{A}_5$, operators $\mathcal{L}$ are considered as operators on spaces:

$$\mathcal{L}:\mathcal{F}_{-1/2}\oplus\mathcal{F}_{1/2}\to\mathcal{F}_{3/2}\oplus\mathcal{F}_{1/2}.$$

- In the case of algebras $\mathcal{A}_6$ and $\mathcal{A}_7$, operators $\mathcal{L}$ are considered as operators on spaces:

$$\mathcal{L}:\mathcal{F}_{-1/2}\oplus\mathcal{F}_{3/2}\to\mathcal{F}_{3/2}\oplus\mathcal{F}_{-1/2}.$$

- In the case of algebra $\mathcal{A}_8$ operators $\mathcal{L}$ are considered as operators on spaces:

$$\mathcal{L}:\mathcal{F}_{-1/2}\oplus\mathcal{F}_{11/2}\to\mathcal{F}_{3/2}\oplus\mathcal{F}_{-9/2}.$$

We will define the action of each of the Lie algebras $\mathcal{A}_i$ on the above spaces of tensor densities. We will show in the next section the relation of this action with the matrix Sturm–Liouville operators (6).

**Proposition-Definition 3.9.** *There exist the following actions of the Lie algebras $\mathcal{A}_i$ on spaces of tensor densities*:

1. $\mathcal{A}_1$ *acts on* $\mathcal{F}_{-1/2} \oplus \mathcal{F}_{1/2}$ *and* $\mathcal{F}_{3/2} \oplus \mathcal{F}_{1/2}$ *via formula* (9) *with* $k = 1$.
2. $\mathcal{A}_2$ *acts on* $\mathcal{F}_{-1/2} \oplus \mathcal{F}_{1/2}$ *and* $\mathcal{F}_{3/2} \oplus \mathcal{F}_{1/2}$ *via formula* (8) *with* $k = 1$ *and* $k = -1$, *respectively*.[1]
3. $\mathcal{A}_3$ *acts on* $\mathcal{F}_{-1/2} \oplus \mathcal{F}_{1/2}$ *and* $\mathcal{F}_{3/2} \oplus \mathcal{F}_{1/2}$ *via formula* (8′) *with*
$$\hat{s}(f, \psi\, \mathrm{d}x^{-1/2}) = -2f'\psi'\, \mathrm{d}x^{1/2} \quad \textit{and} \quad k = 1,$$
$$\hat{s}(f, \psi\, \mathrm{d}x^{1/2}) = -2(f'\psi' + f''\psi)\, \mathrm{d}x^{3/2} \quad \textit{and} \quad k = -1,$$
*respectively*.
4. $\mathcal{A}_4$ *acts on* $\mathcal{F}_{-1/2} \oplus \mathcal{F}_{1/2}$ *and* $\mathcal{F}_{3/2} \oplus \mathcal{F}_{1/2}$ *via formula* (8′) *with*
$$\hat{s}(f, \psi\, \mathrm{d}x^{-1/2}) = -2f\psi'\, \mathrm{d}x^{1/2} \quad \textit{and} \quad k = 1,$$
$$\hat{s}(f, \psi\, \mathrm{d}x^{1/2}) = -2(f\psi' + f'\psi)\, \mathrm{d}x^{3/2} \quad \textit{and} \quad k = -1,$$
*respectively*.
5. $\mathcal{A}_5$ *acts on* $\mathcal{F}_{-1/2} \oplus \mathcal{F}_{1/2}$ *and* $\mathcal{F}_{3/2} \oplus \mathcal{F}_{1/2}$ *via formula* (9′) *with* $k = 1$ *and*
$$\tilde{s}(f, \psi\, \mathrm{d}x^{-1/2}) = f\psi'\, \mathrm{d}x^{1/2},$$
$$\tilde{s}(f, \psi\, \mathrm{d}x^{1/2}) = (f\psi' + f'\psi)\, \mathrm{d}x^{3/2},$$
*respectively*.
6. $\mathcal{A}_6$ *acts on* $\mathcal{F}_{-1/2} \oplus \mathcal{F}_{3/2}$ *and* $\mathcal{F}_{3/2} \oplus \mathcal{F}_{-1/2}$ *via formula* (8′) *with*
$$\hat{s}(f, \psi\, \mathrm{d}x^{-1/2}) = -2f'\psi''\, \mathrm{d}x^{3/2} \quad \textit{and} \quad k = 1,$$
$$\hat{s}(f, \psi\, \mathrm{d}x^{-1/2}) = (2f'\psi'' + 4f''\psi' + 2f'''\psi)\, \mathrm{d}x^{3/2} \quad \textit{and} \quad k = -1,$$
*respectively*.
7. $\mathcal{A}_7$ *acts on* $\mathcal{F}_{-1/2} \oplus \mathcal{F}_{3/2}$ *and* $\mathcal{F}_{3/2} \oplus \mathcal{F}_{-1/2}$ *via formula* (8′) *with*
$$\hat{s}(f, \psi\, \mathrm{d}x^{-1/2}) = -2f\psi''\, \mathrm{d}x^{3/2} \quad \textit{and} \quad k = 1,$$
$$\hat{s}(f, \psi\, \mathrm{d}x^{-1/2}) = (2f\psi'' + 4f'\psi' + 2f''\psi)\, \mathrm{d}x^{3/2} \quad \textit{and} \quad k = -1,$$
*respectively*.
8. $\mathcal{A}_8$ *acts on* $\mathcal{F}_{-1/2} \oplus \mathcal{F}_{11/2}$ *and* $\mathcal{F}_{3/2} \oplus \mathcal{F}_{-9/2}$ *via formula* (9′) *with* $k = 2$, *and respectively*:
$$\tilde{s}(f, \psi\, \mathrm{d}x^{-1/2}) = (5f'''\psi^{(\mathrm{IV})} - 10f^{(\mathrm{IV})}\psi''' + 3f^{(\mathrm{V})}\psi'')\, \mathrm{d}x^{11/2},$$

[1] Note that $\mathcal{A}_1$ and $\mathcal{A}_2$ are just central extensions of the semi-direct product.

$$\tilde{s}(f,\psi\,\mathrm{d}x^{-9/2}) = (-18f^{(\mathrm{VII})}\psi - 56f^{(\mathrm{VI})}\psi' - 63f^{(\mathrm{IV})}\psi'' - 30f^{(\mathrm{IV})}\psi''' \\ -5f'''\psi^{(\mathrm{IV})})\,\mathrm{d}x^{3/2}.$$

**Proof.** Straightforward calculation. □

**Remark.** *The mappings $\tilde{s}$ and $\hat{s}$ can be defined modulo* 1-*cocycles* (*cf. Section* 3.2.2). *However, the considered actions are uniquely defined by their relations to the matrix Sturm–Liouville operators. It can be proven that the defined $\mathcal{A}_i$-actions on the spaces of tensor densities are the unique actions corresponding to the $\mathcal{A}_i$-actions on the spaces of matrix Sturm–Liouville operators*.

### 3.3. Main result: spaces of matrix operator $\mathcal{L}$ as an $\mathcal{A}_i$-module

In this section, we define the action of Lie algebra $\mathcal{A}_i$ on the corresponding space of operators $\mathcal{L}$ from Table 1. This definition is analogous to (4) in the case of *Sturm–Liouville* operators and provides a geometric realization of the coadjoint action of $\mathcal{A}_i$.

**Proposition 3.10.** *Each of the Lie algebra $\mathcal{A}_i$ for $i = 1, \ldots, 8$ acts on the corresponding space of matrix Sturm–Liouville operators as follows*:

$$\mathcal{T}_i^*(\mathcal{L}_i) = \mathcal{T}_i \circ \mathcal{L}_i - \mathcal{L}_i \circ \mathcal{T}_i, \tag{11}$$

*where $\mathcal{T}_i$ are the actions given in Proposition* 3.9.

This proposition is a generalization of Definition 1.1 to the algebras $\mathcal{A}_i$ and operators $\mathcal{L}$ associated.

The main result of this paper is the following.

**Theorem 3.11.** *The action* (11) *coincides with the coadjoint action of the Lie algebra $\mathcal{A}_i$*.

The proof can be obtained by a straightforward calculation.

We hope that such a realization can be useful for the theory of KdV-type integrable systems related to the Lie algebras $\mathcal{A}_i$, for the study of the coadjoint orbits of these algebras, etc. (cf. [6] for the Virasoro case).

## 4. Negative result: case of Lie algebra $\mathcal{A}_9$

The generalization of the Kirillov–Segal result does not hold in the case of algebra $\mathcal{A}_9$. The semi-direct product *Vect*$(S^1) \ltimes \mathcal{F}_7$ acts on $\mathcal{F}_{-1/2} \oplus \mathcal{F}_{13/2}$ by (8) and on $\mathcal{F}_{-1/2} \oplus \mathcal{F}_{15/2}$ by (9), we are thus looking for a deformed action of $\mathcal{A}_9$ on $\mathcal{F}_{-1/2} \oplus \mathcal{F}_{13/2}$ and $\mathcal{F}_{-1/2} \oplus \mathcal{F}_{15/2}$ via formulae (8′) and (9′).

**Proposition 4.1.** *There is no $\mathcal{A}_9$-module structure on spaces $\mathcal{F}_{-1/2} \oplus \mathcal{F}_{13/2}$ and $\mathcal{F}_{-1/2} \oplus \mathcal{F}_{15/2}$ in class of actions* (8′) *and* (9′).

**Proof.** Under the notations of Section 3.2.2, we will show that there is no deformation $\hat{T}_9$ (resp. $\tilde{\mathcal{T}}_9$) of the action of the semi-direct product *Vect*$(S^1) \ltimes \mathcal{F}_7$ on space $\mathcal{F}_{-1/2} \oplus \mathcal{F}_{13/2}$ (resp. $\mathcal{F}_{-1/2} \oplus \mathcal{F}_{15/2}$). Let us give the details in the case of action $\hat{T}_9$ and space $\mathcal{F}_{-1/2} \oplus \mathcal{F}_{13/2}$.

The proof uses the notion of transvectants. Let us first recall the main definitions.

Consider the bilinear mappings on tensor densities: $J_k : \mathcal{F}_\lambda \oplus \mathcal{F}_\mu \to \mathcal{F}_{\lambda+\mu+k}$ with $k$ integer defined by

$$J_k(a\,\mathrm{d}x^\lambda, b\,\mathrm{d}x^\mu) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \prod_{r,s=i}^{k-1} (-2\mu - r)(-2\lambda - s) a^{(k-i)} b^{(i)}. \tag{12}$$

The operations (12) are the so-called Gordan's transvectants [3] (rediscovered by Rankin [13] and Cohen [1] in the theory of modular functions).

Consider the Lie subalgebra of *Vect*$(S^1)$ generated by the vector fields:

$$\frac{\mathrm{d}}{\mathrm{d}x}, x\frac{\mathrm{d}}{\mathrm{d}x}, x^2\frac{\mathrm{d}}{\mathrm{d}x}$$

where $x$ is the affine parameter on $\mathbf{RP^1} \cong S^1$. This subalgebra is isomorphic to $sl_2(\mathbf{R})$. It is well known that for each $k$, the mapping $J_k$ is the unique $sl_2$-equivariant on the tensor densities.

Under the above notations one has the following lemma.

**Lemma 4.2.** *Suppose that the Lie algebra $\mathcal{A}_9$ acts on the space $\mathcal{F}_{-1/2} \oplus \mathcal{F}_{13/2}$. Then the complementary term $\hat{s}(f, \psi)$ from* (8′) *is necessarily $sl_2$-equivariant.*

**Proof.** Consider the commutator on $\mathcal{A}_9$. This commutator can be written with transvectants; exactly we have (up to a constant)

$$c_7(f, g) = J_9(f, g), \quad \left[f(x)\frac{\mathrm{d}}{\mathrm{d}x}, g(x)\frac{\mathrm{d}}{\mathrm{d}x}\right] = J_1(f, g), \quad L_f b = J_1(f, b\,\mathrm{d}x^7).$$

We have the same result for the action (8) from Section 3.2.1 since $\psi a = J_0(\psi, a\}$. Hence Lemma 4.2 is proven. □

The property of *$sl_2$-equivariance* implies that the term $\hat{s}(f, \psi)$ of $\hat{T}_9$ is (up to a constant) $\hat{s}(f, \psi) = J_8(f, \psi)$. Indeed, the transvectant $J_8$ is unique $sl_2$-equivariant map: *Vect*$(S^1) \oplus \mathcal{F}_{-1/2} \to \mathcal{F}_{13/2}$.

**Lemma 4.3.** *The map $\hat{T}_9$ given by* (8′) *with $\hat{s}(f, \psi)$ proportional to $J_8(f, \psi)$ does not define an action of the Lie algebra $\mathcal{A}_9$ on space $\mathcal{F}_{-1/2} \oplus \mathcal{F}_{13/2}$.*

**Proof.** Straightforward calculation. □

We have the same result for $\tilde{\mathcal{T}}_9$ given by (9′) with $\tilde{s}(f, \psi) = J_9(f, \psi)$. Proposition 4.1 follows from Lemmas 4.2 and 4.3. □

**Remark 4.4.** *In the case of algebra* $\mathcal{A}_8$, *Lemma* 4.2 *is still valid, so that the term* $\tilde{s}(f, \psi)$ *given in* (8) *is necessarily* $sl_2$*-equivariant. This is verified, since* (*up to a constant*)

- *in the case of the action on* $\mathcal{F}_{-1/2} \oplus \mathcal{F}_{11/2}$, *one had*

$$\tilde{s}(f, \psi\, \mathrm{d}x^{-1/2}) = (5f'''\psi^{(\mathrm{IV})} - 10f^{(\mathrm{IV})}\psi''' + 3f^{(\mathrm{V})}\psi'')\, \mathrm{d}x^{11/2} = J_7(f, \psi\, \mathrm{d}x^{-1/2});$$

- *in the case of the action on* $\mathcal{F}_{3/2} \oplus \mathcal{F}_{-9/2}$, *one had*

$$\begin{aligned}\tilde{s}(f, \psi) &= (-18f^{(\mathrm{VII})}\psi - 56f^{(\mathrm{VI})}\psi' - 63f^{(\mathrm{IV})}\psi'' \\ &\quad -30f^{(\mathrm{IV})}\psi''' - 5f'''\psi^{(\mathrm{IV})})\, \mathrm{d}x^{-3/2} \\ &= J_7(f, \psi\, \mathrm{d}x^{-9/2}).\end{aligned}$$

But in the case of algebra $\mathcal{A}_8$, $\tilde{T}_8$ is indeed an $\mathcal{A}_8$-action.

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